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Weakly nonlinear surface waves over a random seabed

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We study the effects of multiple scattering of slowly modulated water waves by a weakly random bathymetry. The combined effects of weak nonlinearity, dispersion and random irregularities are treated together to yield a nonlinear Schrödinger equation with a complex damping term. Implications for localization and side-band instability are discussed. Transmission and nonlinear evolution of a wave packet past a finite strip of disorder is examined.

1. Introduction

In areas of classical physics such as electromagnetism, acoustics and seismology, there is extensive literature on the propagation of infinitesimal sinusoidal waves in random media. Based on linearized field equations, perturbation theories have been developed for weak random inhomogeneities (see Karal & Keller 1964; Keller 1964; Chernov 1967; Frisch 1968; Soong 1973; Ishimaru 1997). On the basis of linearized equations, Asch *et al.* (1991) have treated infinitesimal sound pulses in randomly layered media for weak and strong inhomogeneities. They focused on cases where the correlation length is much less than the typical wavelength, which is in turn much less than the extent of the region of randomness. In one-dimensional wave propagation, if the inhomogeneities extend over a large spatial region, multiple scattering yields a complex change in the propagation constant, the real part of which corresponds to a change of wavenumber and the imaginary part to spatial attenuation. In particular, the latter is effective for a broad range of incident wave frequencies and is a distinctive feature of randomness, first found in condensed-matter physics (a conductor with disordered properties can turn into an insulator) by Anderson (1958). This is in sharp contrast to periodic inhomogeneities which cause strong Bragg scattering only for certain frequency bands. A survey of localization theories in many branches of classical physics based on linearized equations can be found in Sheng (1990, 1995).

Of interest to coastal oceanography, the propagation of surface waves over a random seabed in water of intermediate depth (i.e. comparable to the wavelength) has been studied by Hasselman (1966) and Long (1973) using the technique of Feynman diagrams. Similar techniques have also been employed by Elter & Molyneux (1972) to study linearized long tsunami waves propagating across an ocean with a random bathymetry. The laboratory experiments of Belzons, Guazzelli & Parodi (1988) and the companion linear theory of Devillard, Dunlop & Souillard (1988) have aroused interest in the study of the localization of infinitesimal waves over a random bathymetry, because of the oceanographic implications for wave transformation over long distances. Further linearized theories have been reported by Nachbin & Papanicolaou (1992) and Nachbin (1995) for waves over large bathymetric variations with scales comparable to the mean depth. More recently, the linearized problem of weak scattering by small random irregularities on the seabed has been studied by Pelinovsky, Razin & Sasorova (1998), who obtained analytical results for the propagation constant of a simple-harmonic wave train. The same problem has been reinvestigated in Stepaniants (2001) using a diagrammatic technique.

Considerable theoretical advances in nonlinear wave propagation in random media have been made in mathematical physics. In a seminal paper, Devillard & Souillard (1986) have studied the one-dimensional nonlinear Schrödinger equation (NLS) with a random potential. For a stationary wave passing through a random medium of thickness L, they find the transmission coefficient to attenuate exponentially with increasing L if nonlinearity is weak. For sufficiently strong nonlinearity, however, the attenuation is slowed and can become only polynomial. Confirmations and extensions to other random potentials have been given by Doucot & Rammal (1987), Kivshar *et al.* (1990), Gredeskul & Kivshar (1992) and Bronski (1998). Theories for non-stationary incident waves, such as solitons, passing through a random potential have been advanced by many researchers, e.g. Gredeskul & Kivshar (1992), Knapp, Papanicolaou & White (1991), Knapp (1995), Garnier (1998), Garnier (2001*b*). Of particular interest is the finding (Garnier 2001*b*) that the manner of soliton transmission depends on the power spectrum of the random perturbations. A theory for the KdV equation with a weak and random potential has also been studied by Garnier (2001*a*). The review by Bass *et al.* (1988) and the article by Knapp, Papanicolaou & White (1989) are also germane.

Published articles on nonlinear water waves over a randomly irregular seabed are relatively scarce. Howe (1971) and Rosales & Papanicolaou (1983) examined shallow water waves. Since linear theory has so far yielded exponential attenuation in space (localization), it is useful to examine whether nonlinearity alters this conclusion. This point is of oceanographic interest, since such attenuation amounts to an effective dissipation by a conservative mechanism of multiple scattering, unlike bottom friction or wave breaking. As a first step, we study here the effects of random depth variations on nonlinear surface waves with a narrow-frequency band. Attention is limited to two space dimensions (vertical and horizontal) and to narrow-banded waves over a weakly random bottom of constant mean depth. The length scale of the random perturbations, ℓ, is assumed to be comparable to the wavelength $2\pi/k$ and to the mean depth h, all of which are much smaller than the length scale of the wave modulation, $2\pi/\varepsilon k$, where ε is a small parameter characterizing the slope of both the surface waves and the seabed irregularities. The total range of propagation and the extent of the random bathymetry are assumed to be even longer, $\sim 2\pi/\epsilon^2 k$. Following Mei & Pihl (2002), who studied waves on a nonlinear string in elastic surroundings with random properties, we employ the method of multiple scales to treat, in a unified manner, localization and slow modulation due to dispersion and weak nonlinearity. We shall show that the envelope of a narrow-banded wave train is governed by a modified nonlinear Schrödinger equation with an additional linear term. The complex coefficient of the new term is not stochastic, but is the autocorrelation of the random perturbations; it is also of the same order as the other terms in the envelope equation. Therefore, the deductions are expected to be somewhat different than those based on the NLS equation with a stochastic (and weak) potential. Physical implications are explored analytically and numerically, for both infinitesimal and weakly nonlinear waves.

2. Evolution equations of the wave envelope

2.1. *Multiple-scale expansions*

We adopt the usual assumptions of inviscid irrotational flow and consider only two-dimensional motion in the (x, z)-plane. Let the seabed be described by $z = -h + b(x)$, where the mean depth h is constant, but $b(x)$ is a random function of x with zero mean. Let the typical slopes of both the free surface height and the seabed roughness height be small, i.e. $k\zeta \sim kb = O(\varepsilon) \ll 1$. The governing equations and nonlinear boundary conditions for the velocity potential $\phi(x, z, t)$ and the free surface displacement $\zeta(x, t)$ are well known and are not repeated here. To allow for slow modulations due to weak nonlinearity and spatial attenuation, we introduce the multiple-scale variables $x_1 = \varepsilon x$, $x_2 = \varepsilon^2 x, \ldots$; $t_1 = \varepsilon t$, $t_2 = \varepsilon^2 t, \ldots$. The bathymetric variation is assumed to depend on the fast and slow scales, i.e. $b = b(x, x_1, x_2)$. Expanding the velocity potential and free surface height as

$$\phi = \varepsilon\phi_1 + \varepsilon^2\phi_2 + \varepsilon^3\phi_3 + \cdots, \quad \zeta = \varepsilon\zeta_1 + \varepsilon^2\zeta_2 + \varepsilon^3\zeta_3 + \cdots, \tag{2.1}$$

where $\phi_n = \phi_n(x, x_1, x_2, \ldots, z, t, t_1, t_2, \ldots)$ and $\zeta_n = \zeta_n(x, x_1, x_2, \ldots, t, t_1, t_2, \ldots)$, we obtain a sequence of perturbation problems similar to those for the simpler case of a horizontal (deterministic) seabed (Mei 1989). The known results are

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\phi_n = F_n, \quad -h < z < 0, \tag{2.2}$$

from the governing Laplace equation, and

$$\mathscr{L}\phi_n \equiv \left(g\frac{\partial}{\partial z} + \frac{\partial^2}{\partial t^2}\right)\phi_n = G_n, \quad z = 0, \tag{2.3}$$

from the free surface condition (combining kinematic and dynamic requirements). Once the velocity potential is found, the free surface height follows from the dynamic condition of constant pressure,

$$-g\zeta_n = H_n, \quad z = 0. \tag{2.4}$$

The forcing terms at the first two orders are

$$F_1 = 0, \quad F_2 = -2\phi_{1xx_1},$$

$$G_1 = 0, \quad G_2 = -[\zeta_1\mathscr{L}_z\phi_1 + (\phi_{1x}^2 + \phi_{1z}^2)_t + 2\phi_{1tt_1}],$$

$$H_1 = \phi_{1t}, \quad H_2 = \phi_{2t} + \tfrac{1}{2}(\phi_{1x}^2 + \phi_{1z}^2) + \phi_{1t_1} + \zeta_1\phi_{1zt},$$

where the linear operator $\mathscr{L}$ is defined in (2.3). Only the seabed boundary condition needs to be reconsidered:

$$\phi_z - \varepsilon b_x\phi_x = 0, \quad z = -h + \varepsilon b. \tag{2.5}$$

Expanding about the mean seabed, $z = -h$, we have

$$\phi_{1z} + \varepsilon(\phi_{2z} - (b\phi_{1x})_x) + \varepsilon^2(\phi_{3z} - (b\phi_{2x})_x - \tfrac{1}{2}(b^2\phi_{1xz})_x) = 0,$$

on $z = -h$. Equating like powers of ε yields

$$\frac{\partial\phi_n}{\partial z} = I_n, \quad z = -h, \tag{2.6}$$

where

$$I_1 = 0, \quad I_2 = (b\phi_{1x})_x, \quad I_3 = (b\phi_{2x})_x. \tag{2.7}$$

Let $\langle \ldots \rangle$ be the stochastic average (hence deterministic) and $(\ldots)'$ the random component. At all orders, we express the solutions as

$$\phi_n = \langle \phi_n \rangle + \phi_n', \quad \zeta_n = \langle \zeta_n \rangle + \zeta_n', \quad n = 1, 2, 3, \ldots. \tag{2.8}$$

We also write

$$F_n = \langle F_n \rangle + F_n', \quad G_n = \langle G_n \rangle + G_n', \quad H_n = \langle H_n \rangle + H_n', \quad I_n = \langle I_n \rangle + I_n'. \tag{2.9}$$

By definition, the averages of all the random components above vanish. Note that since $F_1 = G_1 = I_1 = 0$, ϕ_1 is not directly affected by randomness at this order so that

$$\phi_1' = \zeta_1' = 0; \quad \phi_1 = \langle \phi_1 \rangle, \quad \zeta_1 = \langle \zeta_1 \rangle, \tag{2.10}$$

2.2. *The mean components at $O(1)$ and $O(\varepsilon)$*

We take the leading-order solution to be a monochromatic wave train propagating from left to right,

$$\phi_1 = \langle \phi_1 \rangle = \phi_{10} + (\phi_{11}\,\mathrm{e}^{\mathrm{i}\psi} + *) = \phi_{10} - \frac{g}{2\omega}\frac{\cosh Q}{\cosh q}(\mathrm{i}A\,\mathrm{e}^{\mathrm{i}\psi} + *), \tag{2.11}$$

$$\zeta_1 = \langle \zeta_1 \rangle = \tfrac{1}{2}A\,\mathrm{e}^{\mathrm{i}\psi} + *, \tag{2.12}$$

where the zeroth harmonic $\phi_{10} = \phi_{10}(x_1, x_2, t_1, t_2, \ldots)$ represents the long-wave potential. A denotes the leading-order wave amplitude, $\psi = kx - \omega t$ the wave phase, $q = kh$, $Q = k(z + h)$ and $*$ denotes the complex conjugate. The dispersion relation

$$\omega^2 = gk \tanh kh \tag{2.13}$$

relates the frequency ω and the wavenumber k.

From (2.7) and (2.10), $\langle I_2 \rangle = (\langle b \rangle \phi_{1x})_x = 0$. Thus, the boundary value problem for $\langle \phi_2 \rangle$ is independent of the bed roughness b and the solution is formally the same as that for a horizontal seabed,

$$\langle \phi_2 \rangle = \phi_{20} - \frac{\omega Q \sinh Q}{2k^2 \sinh q}\left(\frac{\partial A}{\partial x_1}\mathrm{e}^{\mathrm{i}\psi} + *\right) - \frac{3}{16}\frac{\omega \cosh 2Q}{\sinh^4 q}(\mathrm{i}A^2\,\mathrm{e}^{2\mathrm{i}\psi} + *), \tag{2.14}$$

$$\langle \zeta_2 \rangle = -\frac{1}{g}\frac{\partial \phi_{10}}{\partial t_1} - \frac{k|A|^2}{2\sinh 2q} + \frac{k\cosh q(1 + 2\cosh^2 q)}{8\sinh^3 q}(A^2\,\mathrm{e}^{2\mathrm{i}\psi} + *)$$
$$+ \frac{1}{2\omega}\left(\mathrm{i}\frac{\partial A}{\partial t_1}\mathrm{e}^{\mathrm{i}\psi} + *\right) - \frac{q\sinh q}{2k\cosh q}\left(\mathrm{i}\frac{\partial A}{\partial x_1}\mathrm{e}^{\mathrm{i}\psi} + *\right) \tag{2.15}$$

(Mei 1989), where $\phi_{20} = \phi_{20}(x_1, x_2, t_1, t_2, \ldots)$.

Solvability of the first harmonic of $\langle \phi_2 \rangle$ yields the well-known result

$$\frac{\partial A}{\partial t_1} + c_g \frac{\partial A}{\partial x_1} = 0, \tag{2.16}$$

where

$$c_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{\omega}{2k}\left(1 + \frac{2kh}{\sinh 2kh}\right) \tag{2.17}$$

is the group velocity. The phase speed is denoted here as $c = \omega/k$.

2.3. *Random component at $O(\varepsilon)$*

To derive the random components, we use again the fact that $\langle I_2 \rangle = 0$, so that the bottom boundary condition (2.6) gives

$$\frac{\partial \phi_2'}{\partial z} = I_2' = (b\phi_{1x})_x, \quad z = -h. \tag{2.18}$$

It follows that ϕ_2' contains only the first harmonic,

$$\phi_2' = \phi_{21}' \, \mathrm{e}^{-\mathrm{i}\omega t} + *, \quad \zeta_2' = \zeta_{21}' \mathrm{e}^{-\mathrm{i}\omega t} + *. \tag{2.19}$$

The boundary-value problem for ϕ_{21}' is

$$\left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} \right) \phi_{21}' = 0, \quad -h < z < 0, \tag{2.20}$$

$$\left(g \frac{\partial}{\partial z} - \omega^2 \right) \phi_{21}' = 0, \quad z = 0, \tag{2.21}$$

$$\frac{\partial \phi_{21}'}{\partial z} = (b(\phi_{11} \, \mathrm{e}^{\mathrm{i}kx})_x)_x = \frac{gkA}{2\omega \cosh q} (b(x) \, \mathrm{e}^{\mathrm{i}kx})_x, \quad z = -h. \tag{2.22}$$

This problem is solved by using Green's function $\mathscr{G}(x, z\,; x')$, defined by

$$\mathscr{G}_{xx} + \mathscr{G}_{zz} = 0, \quad -h < z < 0, \tag{2.23}$$

$$\mathscr{G}_z - \frac{\omega^2}{g} \mathscr{G} = 0, \quad z = 0, \tag{2.24}$$

$$\mathscr{G}_z = \delta(x - x'), \quad z = -h, \tag{2.25}$$

and the radiation condition that $\mathscr{G}$ behaves as outgoing waves at $\pm\infty$. Relegating the details of $\mathscr{G}$ to Appendix A, we point out that

$$\mathscr{G}(x, z\,; x') = \mathscr{G}\left(|x - x'|, z\right). \tag{2.26}$$

After using Green's theorem, the solution for ϕ_{21}' is found to be

$$\phi_{21}' = \frac{gkA}{2\omega \cosh q} \int_{-\infty}^{\infty} \left(b(x')\mathrm{e}^{\mathrm{i}kx'} \right)_{x'} \mathscr{G}\left(|x - x'|, z\right) \mathrm{d}x', \tag{2.27}$$

which is a random function of x.

2.4. *Mean component at $O(\varepsilon^2)$*

Ensemble-averaging the equations for ϕ_3 gives

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) \langle \phi_3 \rangle = \langle F_3 \rangle, \quad -h < z < 0, \tag{2.28}$$

$$\mathscr{L} \langle \phi_3 \rangle = \langle G_3 \rangle, \quad z = 0, \tag{2.29}$$

$$\frac{\partial \langle \phi_3 \rangle}{\partial z} = \langle I_3 \rangle, \quad z = -h, \tag{2.30}$$

where the operator $\mathscr{L}$ is defined in (2.3). By (2.10), ϕ_1 and ζ_1 are deterministic and hence the forcing functions at $O(\varepsilon^2)$ are given by

$$\langle F_3\rangle = -\left[\phi_{1x_1x_1} + 2\phi_{1xx_2} + 2\langle\phi_2\rangle_{xx_1}\right],$$

$$\begin{aligned}\langle G_3\rangle = -\Big[&\langle\zeta_2\rangle\mathscr{L}_z\phi_1 + \zeta_1\mathscr{L}_z\langle\phi_2\rangle + \tfrac{1}{2}\zeta_1^2\mathscr{L}_{zz}\phi_1 + 2(\phi_{1x}\langle\phi_2\rangle_x + \phi_{1z}\langle\phi_2\rangle_z)_t \\ &+ \zeta_1\left(\phi_{1x}^2 + \phi_{1z}^2\right)_{tz} + \frac{1}{2}\left(\phi_{1x}\frac{\partial}{\partial x} + \phi_{1z}\frac{\partial}{\partial z}\right)\left(\phi_{1x}^2 + \phi_{1z}^2\right) \\ &+ 2\langle\phi_2\rangle_{tt_1} + 2\phi_{1z}\phi_{1zt_1} + 2\phi_{1x_1}\phi_{1xt} + 2\phi_{1x}\phi_{1xt_1} \\ &+ 2\phi_{1x}\phi_{1tx_1} + 2\zeta_1\phi_{1ztt_1} + 2\phi_{1tt_2} + \phi_{1t_1t_1}\Big].\end{aligned}$$

Since ϕ_1, $\langle\zeta_2\rangle$ and $\langle\phi_2\rangle$ are independent of $b(x)$, $\langle F_3\rangle$ and $\langle G_3\rangle$ are formally identical to those for a horizontal seabed (Mei 1989). The bed roughness $b(x)$ only affects $\langle I_3\rangle$. From the last of (2.7), we have, on the mean seabed $z = -h$,

$$\langle I_3\rangle = \langle b\phi_{2x}\rangle_x = \left\langle b\left(\langle\phi_2\rangle_x + \phi'_{2x}\right)\right\rangle_x = \langle b\phi'_{2x}\rangle_x = \langle b\phi'_2\rangle_{xx} - \langle b_x\phi'_2\rangle_x. \tag{2.31}$$

We now add the assumption that the random function $b(x)$ depends on x, x_1 and x_2, but is stationary with respect to the fast coordinate x. The correlation length is assumed to be of the same order as a typical wavelength. The correlation function can then be written as

$$\langle b(x)b(x')\rangle = \sigma^2(x_1, x_2)\gamma(\xi), \tag{2.32}$$

where the correlation coefficient γ is an even and real function of $\xi = x - x'$ only, and the root-mean-square σ may depend on the long scales. Note the following identity:

$$\left\langle\frac{\mathrm{d}(b(x))}{\mathrm{d}x}b(x')\right\rangle = \frac{\partial}{\partial x}\langle b(x)b(x')\rangle = \sigma^2\frac{\mathrm{d}\gamma}{\mathrm{d}\xi}. \tag{2.33}$$

From (2.19), (2.27) and (2.33),

$$\begin{aligned}\langle b\phi'_2\rangle_{xx}|_{z=-h} &= \frac{gkA\mathrm{e}^{-\mathrm{i}\omega t}}{2\omega\cosh kh}\frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty}\frac{\partial}{\partial x'}\left(\langle b(x)b(x')\rangle\,\mathrm{e}^{\mathrm{i}kx'}\right)G\left(|x - x'|, -h\right)\mathrm{d}x' + * \\ &= \frac{gk^3\sigma^2A\mathrm{e}^{\mathrm{i}\psi}}{2\omega\cosh kh}\int_{-\infty}^{\infty}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\gamma\mathrm{e}^{-\mathrm{i}k\xi}\right)G\left(|\xi|, -h\right)\mathrm{d}\xi + *,\end{aligned} \tag{2.34}$$

$$\begin{aligned}\langle b_x\phi'_2\rangle_x|_{z=-h} &= \frac{gkA\mathrm{e}^{-\mathrm{i}\omega t}}{2\omega\cosh kh}\frac{\partial}{\partial x}\int_{-\infty}^{\infty}\frac{\partial}{\partial x'}\left(\left\langle\frac{d(b(x))}{dx}b(x')\right\rangle\mathrm{e}^{\mathrm{i}kx'}\right)G\left(|x - x'|, -h\right)\mathrm{d}x' + * \\ &= -\frac{\mathrm{i}gk^2\sigma^2A\mathrm{e}^{\mathrm{i}\psi}}{2\omega\cosh kh}\int_{-\infty}^{\infty}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{\mathrm{d}\gamma}{\mathrm{d}\xi}\mathrm{e}^{-\mathrm{i}k\xi}\right)G\left(|\xi|, -h\right)\mathrm{d}\xi + *.\end{aligned} \tag{2.35}$$

We now define the coefficient β by

$$\langle I_3\rangle = \mathrm{i}\beta A\cosh kh\,\mathrm{e}^{\mathrm{i}\psi} + *. \tag{2.36}$$

Combining (2.31), (2.34), (2.35) and (2.36) gives

$$\beta(x_1, x_2) = \frac{g\left(k\sigma(x_1, x_2)\right)^2}{2\omega\cosh^2 kh}\int_{-\infty}^{\infty}\left\{\left(\frac{\mathrm{d}}{\mathrm{d}\xi} - \mathrm{i}k\right)^2\gamma\right\}\mathrm{e}^{-\mathrm{i}k\xi}G\left(|\xi|, -h\right)\mathrm{d}\xi. \tag{2.37}$$

The integral in (2.37) is merely a complex constant. The complex coefficient β is evaluated explicitly in Appendix B for the case of a Gaussian correlation.

We separate $\langle \phi_3 \rangle$ into different harmonics:

$$\langle \phi_3 \rangle = \langle \phi_{30} \rangle + \left(\langle \phi_{31} \rangle e^{i\psi} + * \right) + \cdots = \langle \phi_{30} \rangle + \left(e^{i\psi} F(x_2, z, t_2) + * \right) + \cdots, \tag{2.38}$$

where randomness only affects the first harmonic, in view of (2.36). In particular, $\langle \phi_{30} \rangle$ is governed by equations unaffected by the bathymetry, and hence the solvability condition for $\langle \phi_{30} \rangle$ is formally the same as that for a horizontal seabed (see (2.36) in Mei 1989, p. 613), and gives the long-wave equation

$$\frac{\partial^2 \phi_{10}}{\partial t_1^2} - gh \frac{\partial^2 \phi_{10}}{\partial x_1^2} = \frac{\omega^3 \cosh^2 q}{2k \sinh^2 q} \frac{\partial |A|^2}{\partial x_1} - \frac{\omega^2}{4 \sinh^2 q} \frac{\partial |A|^2}{\partial t_1}. \tag{2.39}$$

As for the first harmonic in $\langle \phi_3 \rangle$, we substitute (2.36) and (2.38) into (2.28), (2.29) and (2.30) to obtain the boundary value problem

$$\frac{\partial^2 F}{\partial z^2} - k^2 F = F_{31}, \quad -h < z < 0, \tag{2.40}$$

$$\frac{\partial F}{\partial z} - \frac{\omega^2}{g} F = \frac{1}{g} G_{31}, \quad z = 0, \tag{2.41}$$

$$\frac{\partial F}{\partial z} = i\beta A \cosh kh, \quad z = -h, \tag{2.42}$$

where F_{31} and G_{31} are the complex first-harmonic amplitudes of $\langle F_3 \rangle$ and $\langle G_3 \rangle$, respectively, and are given in Mei (1989, (2.37) and (2.38), p. 613). Since the inhomogeneous boundary-value problem above has a non-trivial homogeneous solution, we invoke the solvability condition (Green's Theorem) to obtain

$$\frac{\partial A}{\partial t_2} + c_g \frac{\partial A}{\partial x_2} - \frac{i\omega q \cosh^2 q}{k^2 \sinh^2 q} \frac{\partial^2 A}{\partial x_1^2} + \frac{i}{2\omega} \frac{\partial^2 A}{\partial t_1^2} + \frac{i\omega k^2 \left(\cosh 4q + 8 - 2 \tanh^2 q \right)}{16 \sinh^4 q} |A|^2 A$$
$$- \frac{ik^2 A}{2\omega \cosh^2 q} \left(\frac{\partial \phi_{10}}{\partial t_1} - \frac{2\omega \cosh^2 q}{k} \frac{\partial \phi_{10}}{\partial x_1} \right) - \frac{iq \sinh q}{k \cosh q} \frac{\partial^2 A}{\partial x_1 \partial t_1} - i\beta A = 0, \tag{2.43}$$

where $q = kh$. The effect of the random topography on the wave envelope is isolated in the last term. The known result for a horizontal (deterministic) seabed is simply (2.43) with $\beta = 0$ (see (2.39) in Mei 1989, p. 614). Finally, by combining (2.16) and (2.43), we obtain

$$\left(\frac{\partial}{\partial t_1} + c_g \frac{\partial}{\partial x_1} \right) A + i\varepsilon \left\{ -\frac{\omega''}{2} \frac{\partial^2 A}{\partial x_1^2} + \frac{\omega k^2 \left(\cosh 4q + 8 - 2 \tanh^2 q \right)}{16 \sinh^4 q} |A|^2 A \right.$$
$$\left. - \left(\frac{k^2}{2\omega \cosh^2 q} \frac{\partial \phi_{10}}{\partial t_1} - k \frac{\partial \phi_{10}}{\partial x_1} \right) A \right\} - \varepsilon i\beta A = 0, \tag{2.44}$$

where

$$\omega'' = \frac{d^2 \omega}{dk^2} = \frac{c_g^2}{\omega} - \frac{\omega}{2k^2} \left(1 + 4k^2 h^2 \frac{\cosh 2kh}{\sinh^2 2kh} \right). \tag{2.45}$$

In summary, we derived the pair of equations (2.39) and (2.44) governing the slow evolution of the short-wave envelope A and the long-wave potential ϕ_{10}. Equations (2.39) and (2.44) can be transformed to the standard NLS form with an additional potential as in Devillard & Souillard (1986) and others, except that the potential term here representing the random bathymetry is deterministic.

3. A steady train of attenuated Stokes waves

As a first application of our theory, we examine the limiting case of a steady wave train. There is no dependence on (x_1, t_1, t_2), so that $A = A(x_2)$. Equation (2.43) reduces to

$$c_g \frac{\partial A}{\partial x_2} + \mathrm{i}\alpha |A|^2 A - \mathrm{i}\beta A = 0, \tag{3.1}$$

where

$$\alpha = \frac{\omega k^2(\cosh 4kh + 8 - 2\tanh^2 kh)}{16\sinh^4 kh} > 0.$$

The solution to (3.1) is a modified Stokes wave exponentially attenuated (localized) in the direction of propagation,

$$A = a_0 \exp\left(-\beta_i x_2/c_g\right) \exp\left(\mathrm{i}\frac{\beta_r x_2}{c_g} + \mathrm{i}\frac{\alpha a_0^2}{2\beta_i}(\exp(-2\beta_i x_2/c_g) - 1)\right), \tag{3.2}$$

where a_0 is the real amplitude at $x_2 = 0$ and β_r, β_i are the real and imaginary parts of β, respectively.

3.1. *Localization length*

From (3.2), the amplitude of A decays exponentially as $|A| = a_0 \exp(-\beta_i x_2/c_g)$. Note that the spatial attenuation is exponential and is independent of nonlinearity. If the extent of disorder is L in the x_2 scale, the amplitude at the transmission end is clearly reduced from the incident amplitude by a factor exponentially diminishing in L. Thus the physical consequence of random scattering here is the same as in the simplest cases of localization, i.e. exponential attenuation in space. This is unlike problems based on the NLS equation with a stochastic potential, in which nonlinearity can change the spatial attenuation pattern from exponential to polynomial (Devillard & Souillard 1986). Our localization distance can be defined by

$$L_{loc} = \frac{c_g}{\varepsilon^2 \beta_i}. \tag{3.3}$$

In Appendix B, β_i is shown in general to be expressible in terms of the Fourier transform of $\gamma(\xi)$, so that (3.3) may be written as

$$\frac{L_{loc}}{h} = \frac{(2kh + \sinh 2kh)^2}{2(\varepsilon\sigma k)^2 k^2 h\,[\hat{\gamma}(0) + \hat{\gamma}(2k)]}, \tag{3.4}$$

where

$$\hat{\gamma}(0) = \int_{-\infty}^{\infty} \gamma(\xi)\,\mathrm{d}\xi, \quad \hat{\gamma}(2k) = \int_{-\infty}^{\infty} \mathrm{e}^{-2\mathrm{i}k\xi}\gamma(\xi)\,\mathrm{d}\xi.$$

The above result (3.4) was first obtained by Pelinovsky *et al.* (1998) by analysing the linearized potential flow problem.

As an example, we consider the Gaussian correlation

$$\gamma(\xi) = \exp\left(-\xi^2/\ell_G^2\right), \quad \text{so that} \quad \hat{\gamma}(2k) = \ell_G\sqrt{\pi}\exp\left(-(k\ell_G)^2\right), \tag{3.5}$$

where ℓ_G is the Gaussian correlation distance. Substituting (3.5) into (3.4) yields

$$\frac{L_{loc}}{h} = \frac{(2kh + \sinh 2kh)^2}{2\sqrt{\pi}\varepsilon^2 kh(\sigma/\ell_G)^2(k\ell_G)^3\left(1 + \exp[-(k\ell_G)^2]\right)}. \tag{3.6}$$

The localization length L_{loc} is plotted in figure 1. Large σ (strong disorder) and large σ/ℓ_G (steep roughness) both lead to short localization distances and fast attenuation.

If the correlation length to depth ratio ℓ_G/h and the steepness σ/ℓ_G of the random topography are held fixed, then L_{loc}/h becomes infinite as $kh \to 0$ and as $kh \to \infty$. Thus long waves are only affected by the mean depth h and not by the relatively short bottom roughness, while short waves do not feel the bottom at all. The smallest L_{loc}/h occurs for some intermediate kh near unity. By minimizing L_{loc}/h with respect to k we find the condition for the smallest L_{loc}/h:

$$2 - \frac{1+\cosh 2kh}{1+\sinh 2kh/(2kh)} - \frac{(k\ell_G)^2}{1+\exp[(k\ell_G)^2]} = 0. \tag{3.7}$$

The product $k\ell_G = 2\pi\ell_G/\lambda$ represents the ratio of correlation length to wavelength. If kh and the mean steepness of the roughness, σ/ℓ_G, are held fixed, then $L_{loc}/h \propto \mathscr{F}(k\ell_G)$ where $\mathscr{F}(x) = x^{-3}(1+\exp(-x^2))^{-1}$ is a monotonically decreasing function for $x > 0$. Therefore, as $k\ell_G$ increases (longer roughness relative to the wavelength), the localization length L_{loc} decreases, indicating stronger attenuation. For $k\ell_G \ll 1$, waves are too long relative to the correlation length to be affected by the random bed roughness. On the other hand, for $k\ell_G \gg 1$, the waves are very short relative to the correlation length and are thus strongly attenuated. These conclusions are similar to those already known for linear waves through a medium with a weakly random index of refraction, e.g. Chen & Soong (1971) and Garnier (2001*b*, p. 151).

It is also interesting that the localization distance is insensitive to the precise form of the correlation function. To show this, we consider the exponential correlation

$$\gamma(\xi) = \exp(-|\xi|/\ell_E), \quad \text{so that} \quad \hat{\gamma}(2k) = \frac{2\ell_E}{1+4(\ell_E k)^2}, \tag{3.8}$$

where ℓ_E is the exponential correlation length. By substituting (3.8) into (3.4), the corresponding localization distance is found to be

$$\frac{L_{loc}}{h} = \frac{(2kh+\sinh 2kh)^2\left(1+4(k\ell_E)^2\right)}{8\varepsilon^2 kh\left(\sigma/\ell_E\right)^2(k\ell_E)^3\left(1+2(k\ell_E)^2\right)}. \tag{3.9}$$

To compare the localization lengths corresponding to the Gaussian and exponential correlations, we choose $\ell_E = \sqrt{\pi}\ell_G/2$ so that (3.8) has the same area as (3.5). With this choice, the localization distances for the two correlations are plotted for the same parameters in figure 1, showing only minor differences.

3.2. *Wavenumber*

In view of (3.2), we find that β contributes to an increase in wavenumber, both directly through β_r and indirectly through β_i which is associated with nonlinearity and amplitude reduction,

$$\Delta k = (\Delta k)_{RD} + (\Delta k)_{NL} \equiv \frac{\varepsilon^2}{c_g}\left(\beta_r - \alpha a_0^2 \exp\left(-\frac{2\beta_i x_2}{c_g}\right)\right). \tag{3.10}$$

Recall that over a strictly horizontal seabed, $\beta_r = \beta_i = 0$ and the wavenumber shift is a constant, corresponding to a Stokes wave. Since $\alpha > 0$, nonlinearity contributes to the reduction of k and hence increases the wavelength. Since the amplitude decays in space, this contribution diminishes with propagation distance. Randomness contributes more directly to the change in wavenumber via $(\Delta k)_{RD}$. In Appendix B, § B.1, β_r is found for the Gaussian correlation. Figure 2 shows that the corresponding $(\Delta k)_{RD}$ is positive for all kh, and hence randomness shortens the wavelength. Since $dc_g/dk = \omega'' < 0$ and $dc/dk = (c_g - c)/k < 0$, randomness also reduces the group and

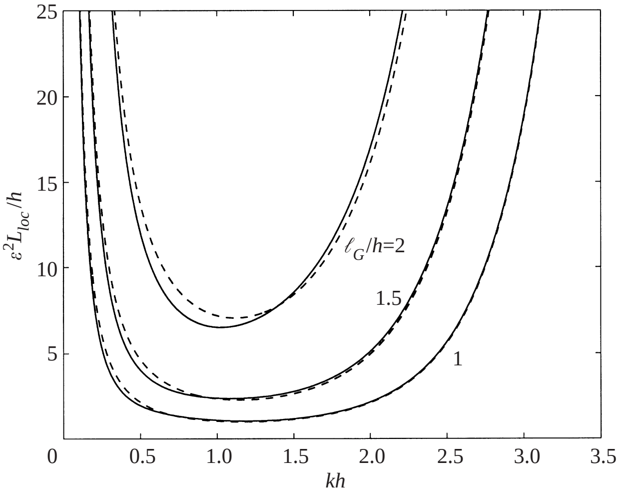

FIGURE 1. Localization length to depth ratio $\varepsilon^2 L_{loc}/h$ corresponding to the Gaussian (solid, (3.6)) and exponential (broken, (3.9)) correlations, for fixed roughness steepness $\sigma/\ell_G = 1$ and various ℓ_G/h. We have chosen $\ell_E = \ell_G\sqrt{\pi}/2$ so that the first moments (areas) of the exponential and Gaussian correlations are the same.

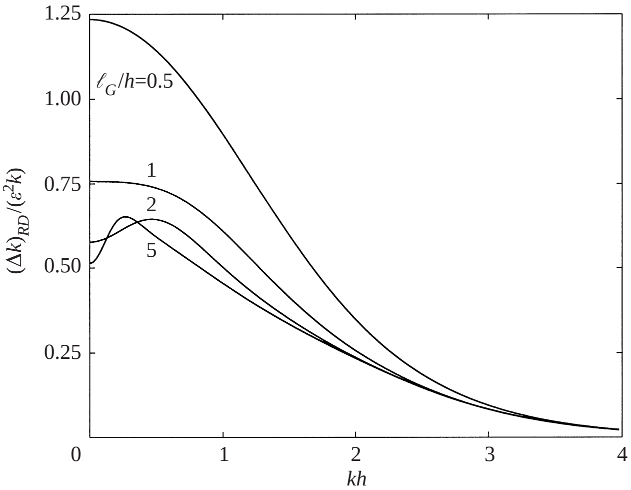

FIGURE 2. Normalized increase in wavenumber $(\Delta k)_{RD}/(\varepsilon^2 k)$ corresponding to the Gaussian correlation, for fixed roughness height $\sigma/h = 1$ and various ℓ_G/h.

phase speeds. Since $\sigma/h = 1$ is fixed, decreasing ℓ_G/h is equivalent to increasing σ/ℓ_G, implying steeper random roughness, which is seen to shorten the waves. For fixed roughness (σ, ℓ_G), $(\Delta k)_{RD}$ decreases with increasing kh in general, since short waves are less affected by the bottom.

3.3. *Comments on other works on linearized waves*

Belzons *et al.* (1988) performed experiments on the localization of infinitesimal water waves over a random bathymetry in a small wave flume of length 4 m and mean water depth h in the range 1 to 4 cm. Bathymetric irregularities were represented by 58 discontinuous steps of random length and amplitude. The step height and step length were uniformly distributed, respectively, between $-\Delta h$ and Δh (zero mean), and between $\ell_B - \Delta\ell$ and $\ell_B + \Delta\ell$. The main results for localization were reported for $h = 1.75$ cm, $\Delta h = 1.25$ cm, $\ell_B = 4.1$ cm and $\Delta\ell = 2.0$ cm. Thus, the height of the steps was not small compared to the mean depth. By definition of b and σ, the dimensional root-mean-square height of the random steps is

$$\varepsilon\sigma = \frac{\Delta h}{\sqrt{3}}. \tag{3.11}$$

It can be shown that the dimensionless correlation coefficient for the random-step bathymetry is

$$\gamma(\xi) = \begin{cases} 1 - \dfrac{|\tau|}{\ell_B}, & 0 \leqslant |\tau| < \ell_B - \Delta\ell, \\ \dfrac{\left(\ell_B + \Delta\ell - |\tau|\right)^2}{4\,\ell_B\,\Delta\ell}, & \ell_B - \Delta\ell \leqslant |\tau| \leqslant \ell_B + \Delta\ell, \\ 0, & |\tau| > \ell_B + \Delta\ell \end{cases} \tag{3.12}$$

(e.g. Stepaniants 2001). The corresponding localization distance is calculated from (3.4), (3.11) and (3.12),

$$\frac{L_{loc}}{h} = \frac{3\ell_B(2kh + \sinh 2kh)^2}{h(k\Delta h)^2}\left(1 + 2(k\ell_B)^2 + \tfrac{2}{3}(k\Delta\ell)^2 - \frac{\sin(2k\Delta\ell)}{2k\Delta\ell}\cos(2k\ell_B)\right)^{-1}. \tag{3.13}$$

Comparison of our theory, (3.13), with the experiments of Belzons *et al.* (1988) produces qualitative agreement. Since the recorded data on the localization length exhibit very large scatter, due in part to averaging over several realizations of the random bed and in part to vortex shedding at the step corners, the comparison is inconclusive and is not presented. Decisive checks must await new experiments for small-amplitude randomness, common in many oceanographic situations.

Devillard *et al.* (1988) and Nachbin (1995) have derived theories for linear gravity waves over large-amplitude random depth variations. Devillard *et al.* (1988) invoked the wide-spacing approximation by neglecting the effects of evanescent modes to predict the localization length. Nachbin (1995) used the results of a numerical Schwarz–Christoffel transformation in a formula for the localization length,

$$\frac{L_{loc}}{\ell} = c_N\left(\frac{\lambda}{\ell}\right)^2, \tag{3.14}$$

where λ is the wavelength, ℓ is the correlation length of the random bed and c_N is a constant computed numerically from Monte Carlo simulations for a set of topographic profiles, for $\lambda/\ell \geqslant 5$. In figure 3, the numerical predictions by Devillard *et al.* (1988) and Nachbin (1995) of the localization distance corresponding to the experiments of Belzons *et al.* (1988) are compared to our formula for small disorder, (3.13). For relatively long waves, there is some qualitative agreement, despite the different realms of intended validity.

Stepaniants (2001) used diagrammatic techniques to study the linearized problem of wave propagation over a random topography with small bathymetric variations. Although the problem is the same as that solved by Pelinovsky *et al.* (1988), a different localization distance was obtained, and is probably in error.

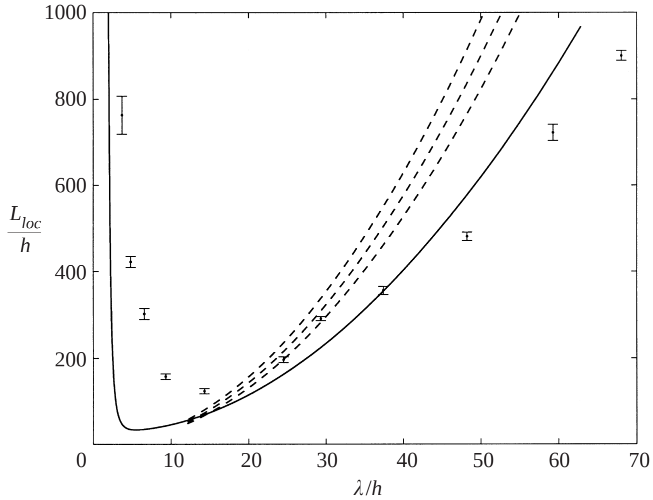

FIGURE 3. Theoretical dependence of localization length L_{loc} on wavelength λ. The dots with error bars represent the theoretical predictions of Devillard *et al.* (1988). The dash-dot lines represent the localization distance (3.14) of Nachbin (1995) with $\ell = \ell_B$ corresponding to $c_N = 0.899, 0.823, 0.753$, from left to right. The solid line represents (3.13) with parameter values $\Delta h/h = 5/7$, $\ell_B/h = 16/7$ and $\Delta\ell/\ell_B = 1/2$.

We next explore the effects of random scattering on the nonlinear evolution of weakly nonlinear waves.

4. Nonlinear evolution

Following the standard procedure (see e.g. Mei 1989, pp. 614–616), (2.39) and (2.44) can be combined to yield

$$\left(\frac{\partial}{\partial t_1} + c_g \frac{\partial}{\partial x_1}\right) A + \mathrm{i}\varepsilon \left\{ \frac{\omega \alpha_1}{k^2} \frac{\partial^2 A}{\partial x_1^2} + \omega k^2 \alpha_2 |A|^2 A + k\left(1 + \frac{c_g}{2c\cosh^2 q}\right) S(t_1) A - \beta A \right\} = 0, \quad (4.1)$$

where $S(t_1)$ is an arbitrary function of time and the dimensionless quantities α_n are given by

$$\alpha_1 = -\frac{\omega''}{2\omega/k^2} = -\frac{1}{2}\frac{c_g^2}{c^2} + \frac{1}{4} + \frac{q^2 \cosh(2q)}{\sinh^2(2q)} > 0, \quad (4.2)$$

$$\alpha_2 = \frac{\cosh 4q + 8 - 2\tanh^2 q}{16 \sinh^4 q} - \frac{\left(2\cosh^2 q + c_g/c\right)^2}{2\sinh^2(2q)\left(q/\tanh q - c_g^2/c^2\right)}. \quad (4.3)$$

Note that the dimensionless coefficients α_1, α_2 are real and β_r, β_i have the dimensions of $1/\text{time}$. Recall the classical result that α_2 is monotonic in kh and is positive (negative) if $kh > (<)\ 1.37$.

Making the transformation

$$A = \varepsilon^{-1} A' \exp\left\{ -\mathrm{i}\varepsilon k \left(1 + \frac{c_g}{2c\cosh^2 q}\right) \int S(t_1)\,\mathrm{d}t_1 + \mathrm{i}\beta_r \varepsilon t_1 \right\} \quad (4.4)$$

and returning to the natural coordinates x, t, (4.1) becomes, in physical variables

$$-\mathrm{i}\left(\frac{\partial}{\partial t}+c_g\frac{\partial}{\partial x}\right)A'+\frac{\omega\alpha_1}{k^2}\frac{\partial^2 A'}{\partial x^2}+\omega k^2\alpha_2\left|A'\right|^2A'-\mathrm{i}\widehat{\beta}_i\left(k\sigma_0\right)^2A'=0, \tag{4.5}$$

where σ_0 is the dimensional root-mean-square bottom roughness height and $\widehat{\beta}_i = \beta_i/(k\sigma)^2$ depends only on kh and ℓ_G/h. We transform to moving coordinates and introduce the dimensionless variables

$$B = A'/A_0, \quad X = k^2A_0\left(x-c_gt\right)\sqrt{\left|\alpha_2\right|/\alpha_1}, \quad \tau = \left|\alpha_2\right|\left(kA_0\right)^2\omega t. \tag{4.6}$$

Equation (4.5) becomes the nonlinear Schrödinger equation with damping†

$$-\mathrm{i}\frac{\partial B}{\partial\tau}+\frac{\partial^2 B}{\partial X^2}+\frac{\alpha_2}{|\alpha_2|}\left|B\right|^2B-\mathrm{i}\Theta B=0, \tag{4.7}$$

where

$$\Theta = \frac{\widehat{\beta}_i}{|\alpha_2|}\left(\frac{\sigma_0}{A_0}\right)^2 \tag{4.8}$$

signifies the relative importance of random and nonlinear effects and can be of order unity.

As is the case for the classical Stokes wave, we have checked numerically from (4.7) that a nonlinear soliton envelope is also exponentially localized over a random region of finite length. Specifically, energy in the transmitted wave packet (no longer a soliton) is reduced from the initial soliton energy by a factor which diminishes exponentially with the length of the random region. This is to be expected, as it is known theoretically (Ablowitz & Segur 1981) from (4.7) that over a random bottom of infinite extent,

$$\frac{\mathrm{d}E}{\mathrm{d}\tau} = -2\,\Theta E, \tag{4.9}$$

where

$$E(\tau) = \int_{-\infty}^{\infty}\left|B(X,\tau)\right|^2\mathrm{d}X \tag{4.10}$$

is the total wave energy in the wave packet.

4.1. *Stokes waves disturbed by side bands*

The special solution of (4.7) uniform in X is equivalent to (3.2),

$$B_S = \exp\left(-\Theta\tau+\mathrm{i}\frac{\alpha_2}{2|\alpha_2|\Theta}\left(\mathrm{e}^{-2\Theta\tau}-1\right)\right). \tag{4.11}$$

In the moving frame of reference, the amplitude decays in time.

Let us first examine how B_S reacts initially to side-band disturbances, and substitute $B = B_S(1+\mathscr{B}')$ into (4.7). Retaining first-order terms in $\mathscr{B}'$, we obtain

$$-\mathrm{i}\frac{\partial\mathscr{B}'}{\partial\tau}+\frac{\partial^2\mathscr{B}'}{\partial X^2}+\frac{\alpha_2}{|\alpha_2|}\mathrm{e}^{-2\Theta\tau}\left(\mathscr{B}'+\mathscr{B}'^*\right)=0. \tag{4.12}$$

† This equation has been studied analytically for weak damping ($\Theta \ll 1$) of soliton envelopes in Ablowitz & Segur (1981) and Fabrikant & Stepanyants (1998).

Substituting $\mathscr{B}' = R + \mathrm{i}I$ into (4.12) and separating real and imaginary parts, we obtain

$$\frac{\partial R}{\partial \tau} - \frac{\partial^2 I}{\partial X^2} = 0, \tag{4.13}$$

$$\frac{\partial I}{\partial \tau} + \frac{\partial^2 R}{\partial X^2} + \frac{2\alpha_2}{|\alpha_2|}\,\mathrm{e}^{-2\Theta\tau} R = 0. \tag{4.14}$$

For a spatially sinusoidal disturbance with modulational wavenumber K,

$$R = \mathrm{Re}\left(\bar{R}(\tau)\,\mathrm{e}^{\mathrm{i}KX}\right), \quad I = \mathrm{Re}\left(\bar{I}(\tau)\,\mathrm{e}^{\mathrm{i}KX}\right). \tag{4.15}$$

Equations (4.13) and (4.14) can be combined to give

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left\{ \begin{array}{c} \bar{R} \\ \bar{I} \end{array} \right\} - K^2\left(\frac{2\alpha_2}{|\alpha_2|}\,\mathrm{e}^{-2\Theta\tau} - K^2\right)\left\{ \begin{array}{c} \bar{R} \\ \bar{I} \end{array} \right\} = 0. \tag{4.16}$$

Instability is possible initially only if $\alpha_2 > 0$, corresponding to deep water with $kh > 1.37$, as in the case without disorder. However, since the carrier wave B_S decays in time, the side band is unstable only if

$$K\sqrt{2\mathrm{e}^{-2\Theta\tau} - K^2} > \Theta. \tag{4.17}$$

Thus, over a random seabed, both the range of instability and the growth rate diminish in the course of propagation. Clearly, if Θ is large, attenuation takes over quickly and an initially unstable side band is unlikely to grow significantly. However, if the randomness is weak relative to nonlinearity, nonlinear effects can still be important for some time.

As an example, we have solved an initial-value problem for the NLS equation (4.7) with $\alpha_2 > 0$ subject to periodic boundary conditions, by a finite difference scheme (Yue & Mei 1980). At $\tau = 0$, the wave envelope contains a carrier wave and a pair of small, symmetric side bands,

$$B(0, X) = 1 + \delta\,\frac{1 - \mathrm{i}}{\sqrt{2}}\cos X, \tag{4.18}$$

where $\delta \ll 1$ is a constant. Numerical results are shown in figure 4, for a case of strong nonlinearity relative to randomness, $\Theta = 0.075$. Here, the wavenumber of the side bands is taken to be 1, which maximizes the left-hand side of (4.17). It can be seen that unstable side bands grow and then oscillate as they exchange energy with the carrier wave. However, over longer times, both the side bands and the carrier wave decay due to random scattering. For larger values of Θ, monotonic decay due to radiation damping dominates the evolution after a short time. Indeed, for $K = 1$, (4.17) implies that instability occurs only when

$$\tau \leqslant \frac{1}{\Theta}\log\frac{2}{\Theta^2 + 1}. \tag{4.19}$$

Thus, monotonic decay begins at $\tau = 0$ if $\Theta \geqslant 1$, and at $\tau \approx 1$ if $\Theta \approx 0.5$.

4.2. *Effects of a finite strip of disorder on a wave packet*

Over a deep and horizontal seabed, it is well known from inverse scattering theory that if the wave envelope is initially a packet of the form

$$B(0, X) = \mathrm{sech}\left(\frac{X}{\sqrt{2M'}}\right), \tag{4.20}$$

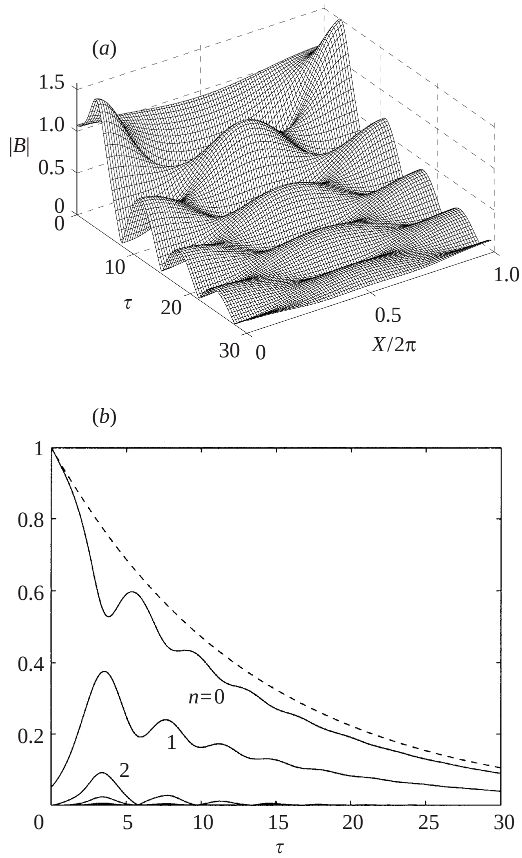

FIGURE 4. Nonlinear evolution of the wave envelope with initial condition (4.18) for $\alpha_2 > 0$, $\Theta = 0.075$ and $\delta = 0.1$. (*a*) Wave envelope amplitude $|B(X,\tau)|$ as a function of the moving coordinate X and normalized time τ. (*b*) Time evolution of $|\hat{B}_n(\tau)|$, where $|\hat{B}_0|$ is the carrier wave amplitude and $|\hat{B}_n|$ the nth harmonic side-band amplitude, for $n \geqslant 1$. The broken line shows the evolution of the amplitude $|B_S|$ of the uniform Stokes wave (4.11).

then M bound solitons will evolve where M is the largest integer less than M' (Satsuma & Yajima 1974). These bound solitons travel together, but exchange energy periodically; the number of distinct modulational periods is $M-1$.

By solving (4.7) numerically we now examine the passage and subsequent evolution of such a packet over a random strip of finite length. Because of the coordinate transformation (4.6), the random strip appears in the (X,τ)-plane as a band inclined to the left at a small slope of $S = O(kA_0)$. For illustration, we choose L, Θ, S such that after transmission, the total energy is reduced to one quarter of its initial value. This corresponds approximately to the envelope height being reduced by half. Specifically, we take $L = 50$, $\Theta = \log 2$ and $S = -1/50$, so that the total duration of the passage is $\Delta\tau = 1$. At $\tau = 0$, the random patch begins at $X = 10$. Figure 5 demonstrates the evolution of the initial wave packet (4.20) with $M' = 2, 4, 6$. Passage over the random patch can be identified by the relatively white region. After passage, the envelope undergoes periodic modulations with 0, 1 and 2, i.e. $M/2-1$, periods, implying the presence of 1, 2 and 3 bound solitons, respectively. This is confirmed by comparing the computed profiles with theoretical bound soliton profiles. Calculations for an initial

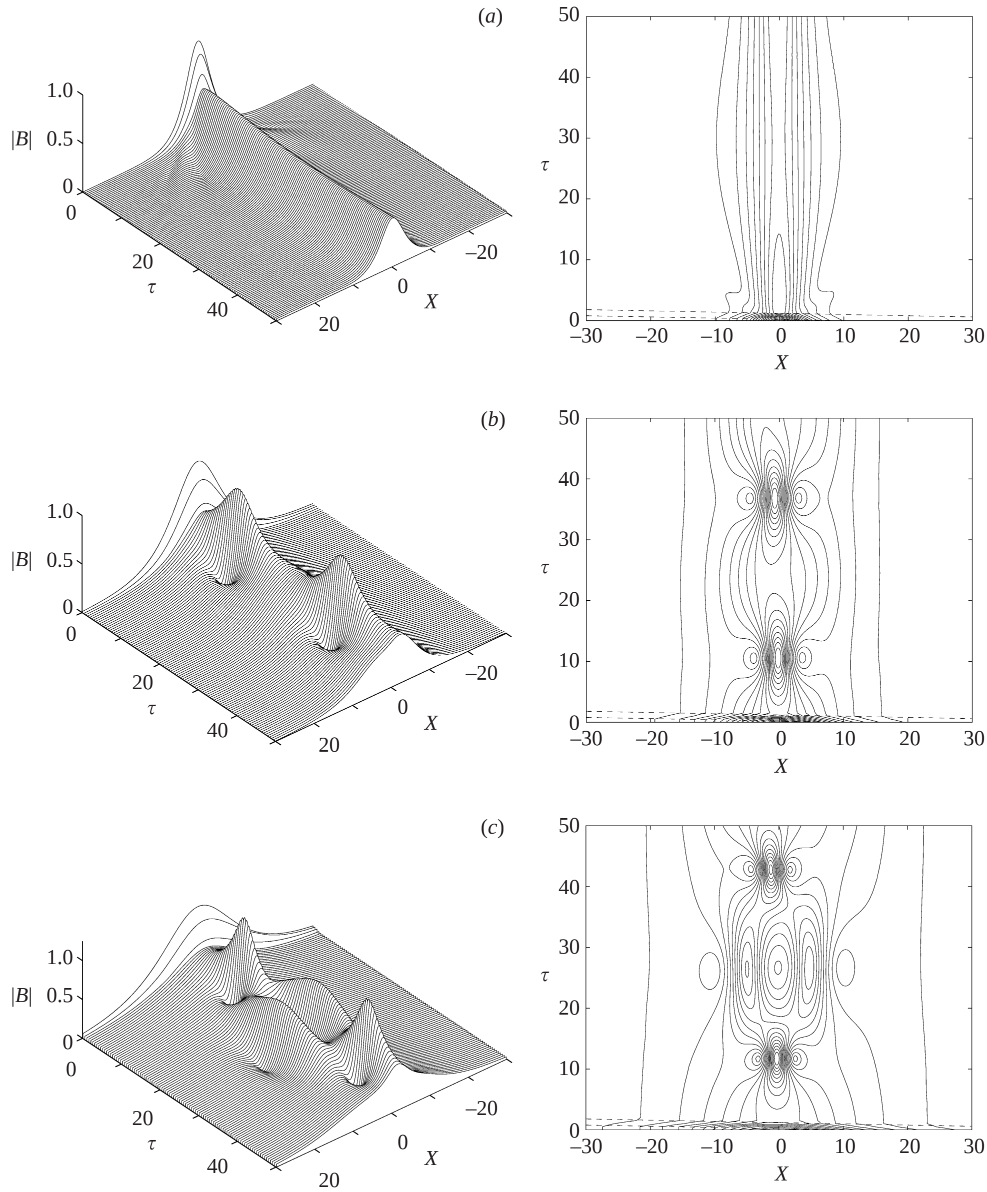

FIGURE 5. Evolution of the wave envelope $|B|$ over a patch of randomness of length $L = 50$, with $\alpha_2 > 0$ and $\Theta = \log 2$. The random patch appears as a slanted strip (broken lines) of slope $S = -1/50$ in the (X, τ)-plane. At $\tau = 0$, the random patch begins at $X = 10$ and the wave envelopes are given by (4.20) with (a) $M' = 2$, (b) $M' = 4$ and (c) $M' = 6$. After crossing the random patch, the envelopes are close to $M'/2$-solitons. The plots in the left column show the wave envelope and those in the right show its contours.

soliton, i.e. $M' = 1$, show that the peak amplitude is reduced by half when leaving the random patch. The envelope flattens out due to dispersion, consistent with the analytical theory; plots are therefore omitted.

To see how different entry times affect the subsequent evolution, we modify the problem corresponding to figure 5(b) so that the same initial wave packet enters the

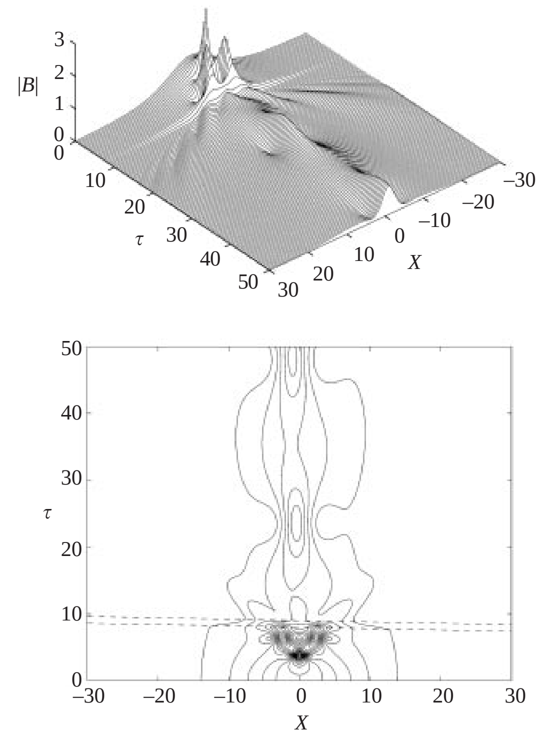

FIGURE 6. The same case as figure 5(b), except that the wave envelope meets the random patch at a later stage in its evolution. At $\tau = 0$, the random patch begins at $X = 400$.

region of disorder only after $\tau \approx 7.5$ (at $\tau = 0$, the random patch begins at $X = 400$). As shown in figure 6, four bound solitons, characterized by two peaks, have already developed before meeting the random region. After passage, still just two bound solitons emerge, suggesting that the main effect of disorder is simply exponential attenuation, beyond which nonlinear effects prevail according to the undamped NLS theory.

As another numerical example involving more intricate physics, we consider a bi-soliton crossing the same random strip. In the absence of disorder, the following solution is known to represent a bi-soliton envelope:

$$B(X, \tau) = g/f, \tag{4.21}$$

where

$$g = E_1(1 + b_2|E_2|^2) + E_2(1 + b_1|E_1|^2), \quad E_m = \exp(k_m X - \mathrm{i}k_m^2\tau + d_m),$$

$$f = 1 + f_1|E_1|^2 + f_2|E_2|^2 + 2\mathrm{Re}\left(\frac{E_1E_2^*}{2\left(k_1 + k_2^*\right)^2}\right) + \frac{1}{64}\left|\frac{k_1 - k_2}{k_1 + k_2^*}\right|^4 \frac{|E_1|^2|E_2|^2}{(\mathrm{Re}\,(k_1)\,\mathrm{Re}\,(k_2))^2},$$

$$f_m = \frac{1}{8(\mathrm{Re}(k_m))^2}, \quad b_m = \frac{(k_1 - k_2)^2}{8(\mathrm{Re}(k_m))^2(k_m^* + k_n)^2} \quad (n = 1, 2;\ n \neq m)$$

(e.g. Johnson 1997, p. 320), where $k_m = a_m/\sqrt{2} + \mathrm{i}c_m/2$ and a_m, c_m are real. When $c_1 = c_2$, the two bound solitons move at the same velocity in the moving coordinate system and exchange energy periodically. For the parameter values $k_1 = 2/\sqrt{2}$, $k_2 = 1.95/\sqrt{2}$, $d_1 = 5$ and $d_2 = 0$, the evolution of the initial wave envelope (4.21) is

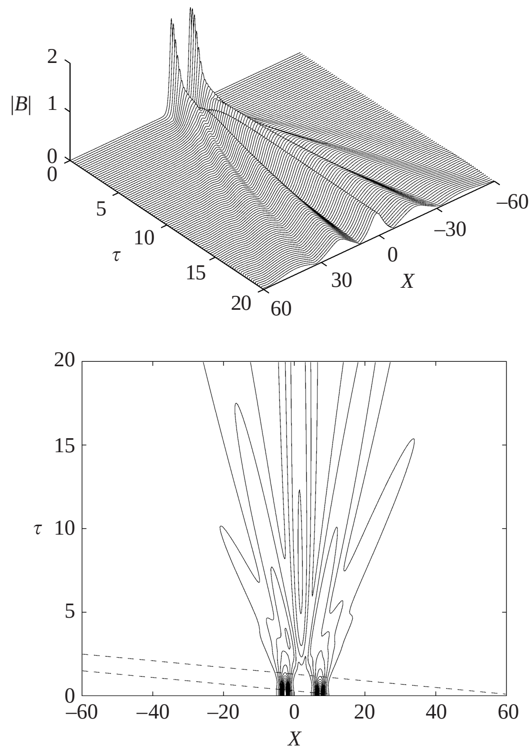

FIGURE 7. Evolution of the wave envelope over the same random patch as that in figure 5. At $\tau = 0$, the wave envelope is the bi-soliton given by (4.21) with $k_1 = 2/\sqrt{2}$, $k_2 = 1.95/\sqrt{2}$, $d_1 = 5$ and $d_2 = 0$. In the absence of randomness, the wave envelope would propagate as a bi-soliton whose crests remain nearly intact and a fixed distance apart.

solved numerically from (4.7) and shown in figure 7. As the twin peaks cross the random patch, their energy is reduced. After leaving the patch, the twin peaks expand in width and interact, causing new and smaller peaks to form. The central peak emerges as a soliton, while those on the sides eventually flatten out due to dispersion. This example serves to show that attenuation due to random scattering can cause drastic changes in some nonlinear waves.

5. Final remarks

In conclusion, we have shown that randomness on the sea bottom (i.e. in the bottom boundary condition) leads to a deterministic wave envelope equation of NLS form. Physically, multiple scattering gives rise to radiation damping which is proportional to the correlation of the randomness. In simple situations, such as a uniform Stokes wave train or a wave packet, the consequence is similar to that known in linearized wave theories, i.e. exponential attenuation in space. This is different from existing theories based on the NLS equation with a random potential, where localization is not necessarily exponential. However, in more complex situations, such as a bi-soliton, unexpected evolutions result due to the interplay between damping and nonlinearity.

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Appendix A. Green's function

Taking the exponential Fourier transform of (2.23), (2.24) and (2.25), it is straightforward to find

$$G(x,z) = \frac{1}{2\pi}\int_C \mathrm{d}\alpha\, \mathrm{e}^{\mathrm{i}\alpha(x-x')} \frac{-(\omega^2/g)\sinh \alpha z - \alpha\cosh\alpha z}{\alpha(\alpha\sinh\alpha h - (\omega^2\cosh\alpha h/g))}. \tag{A 1}$$

To satisfy the radiation condition, we take the integration path C to be the real axis of the complex α-plane, but indented above the real pole at $\alpha = -k$ and below another real pole at $\alpha = k$, where k is the positive real root of the dispersion relation (2.13). The integrand also has imaginary poles at $\pm \mathrm{i}k_n$ which are the positive real roots of

$$\omega^2 = g\mathrm{i}k_n \tanh \mathrm{i}k_n h = -gk_n \tan k_n h, \quad n = 1, 2, 3, \ldots \tag{A 2}$$

By residue calculus it can be shown that, at $z = -h$,

$$G\left(|\xi|, -h\right) = -\frac{\mathrm{i}(\omega^2/gk)\,\mathrm{e}^{\mathrm{i}k|\xi|}}{\omega^2 h/g + \sinh^2 kh} - \sum_n \frac{(\omega^2/gk_n)\,\mathrm{e}^{-k_n|\xi|}}{\omega^2 h/g - \sin^2 k_n h}. \tag{A 3}$$

Appendix B. The coefficient β

Substituting the Green's function (A 3) into (2.37) yields

$$\frac{\beta}{\omega} = \frac{(k\sigma)^2}{2\cosh^2 kh}\left\{ \frac{\mathscr{I}_0}{\omega^2 h/g + \sinh^2 kh} + \sum_{n=1}^{\infty} \frac{k}{k_n}\frac{\mathscr{I}_n}{\omega^2 h/g - \sin^2 k_n h} \right\}, \tag{B 1}$$

where

$$\mathscr{I}_0 = -\frac{\mathrm{i}}{k}\int_{-\infty}^{\infty} \left\{ \left(\frac{\mathrm{d}}{\mathrm{d}\xi} - \mathrm{i}k\right)^2 \gamma \right\} \mathrm{e}^{-\mathrm{i}k\xi + \mathrm{i}k|\xi|}\,\mathrm{d}\xi, \tag{B 2}$$

$$\mathscr{I}_n = -\frac{1}{k}\int_{-\infty}^{\infty} \left\{ \left(\frac{\mathrm{d}}{\mathrm{d}\xi} - \mathrm{i}k\right)^2 \gamma \right\} \mathrm{e}^{-\mathrm{i}k\xi - k_n|\xi|}\,\mathrm{d}\xi. \tag{B 3}$$

Note that $\frac{1}{2}(2n-1)\pi < k_n h < n\pi$, and as $n \to \infty$, $k_n h \sim n\pi$. Thus $\lim_{n\to\infty} \sin k_n h = 0$.

The following properties of $\gamma(\xi)$ are assumed: $\gamma(0) = 1$; $\gamma(\xi)$ is even, real and either has compact support or decays exponentially as $|\xi| \to \infty$; and $\gamma(\xi)$ is twice differentiable everywhere, including the origin. Under these assumptions, (B 2) and (B 3) simplify to

$$\mathscr{I}_0 = \mathrm{i}k \int_0^{\infty} \left(\mathrm{e}^{2\mathrm{i}k\xi} + 1\right)\gamma(\xi)\,\mathrm{d}\xi + 2, \tag{B 4}$$

$$\mathscr{I}_n = -\frac{2}{k}\mathrm{Re}\left\{ \int_0^{\infty} \mathrm{e}^{-(k_n + \mathrm{i}k)\xi} \left(\frac{\mathrm{d}}{\mathrm{d}\xi} - \mathrm{i}k\right)^2 \gamma(\xi)\,\mathrm{d}\xi \right\}. \tag{B 5}$$

Notice that $\mathscr{I}_n$ is real. Equations for β_r and β_i are obtained by substituting (B 4) and (B 5) into the real and imaginary parts, respectively, of (B 1):

$$\frac{\beta_r}{\omega} = \frac{(k\sigma)^2}{2\cosh^2 kh}\left\{\frac{\mathrm{Re}\,(\mathscr{I}_0)}{\omega^2 h/g + \sinh^2 kh} + \sum_{n=1}^{\infty}\frac{k}{k_n}\frac{\mathscr{I}_n}{\omega^2 h/g - \sin^2 k_n h}\right\}, \tag{B 6}$$

$$\frac{\beta_i}{\omega} = \frac{(k\sigma)^2}{2\cosh^2 kh}\frac{\mathrm{Im}\,(\mathscr{I}_0)}{\omega^2 h/g + \sinh^2 kh}. \tag{B 7}$$

Taking the real and imaginary parts of (B 4) gives

$$\mathrm{Re}(\mathscr{I}_0) = 2 - k\int_0^{\infty}\gamma(\xi)\sin(2k\xi)\,\mathrm{d}\xi,$$

$$\begin{aligned}\mathrm{Im}(\mathscr{I}_0) &= k\int_0^{\infty}\gamma\cos(2k\xi)\,\mathrm{d}\xi + k\int_0^{\infty}\gamma\,\mathrm{d}\xi = \frac{k}{2}\int_{-\infty}^{\infty}\gamma\cos(2k\xi)\,\mathrm{d}\xi + \frac{k}{2}\int_{-\infty}^{\infty}\gamma\,\mathrm{d}\xi \\ &= \frac{k}{2}\int_{-\infty}^{\infty}\gamma(\xi)\,\mathrm{e}^{-2\mathrm{i}k\xi}\,\mathrm{d}\xi + \frac{k}{2}\int_{-\infty}^{\infty}\gamma(\xi)\,\mathrm{d}\xi = \frac{k}{2}(\hat{\gamma}(2k) + \hat{\gamma}(0)),\end{aligned}$$

where $\hat{\gamma}(k)$ is the Fourier transform of $\gamma(\xi)$. Hence (B 7) can be rewritten as

$$\frac{\beta_i}{\omega} = \frac{(k\sigma)^2 k(\hat{\gamma}(2k) + \hat{\gamma}(0))}{4\cosh^2 kh(\omega^2 h/g + \sinh^2 kh)}. \tag{B 8}$$

We now consider the Gaussian correlation function.

B.1. *Gaussian correlation*

Substituting $\gamma(\xi) = \exp[-\xi^2/\ell_G^2]$ into (B 4) gives

$$\mathscr{I}_0 = \frac{\mathrm{i}\sqrt{\pi}k\ell_G}{2}(1 + \exp[-(k\ell_G)^2]) + \frac{\sqrt{\pi}k\ell_G}{2}\exp[-(k\ell_G)^2]\mathrm{erfi}(k\ell_G) + 2, \tag{B 9}$$

$$\mathscr{I}_n = \frac{k_n}{k}\mathrm{Re}\left\{2 - \sqrt{\pi}k_n\ell_G\exp\left(\frac{\ell_G^2}{4}(k_n + \mathrm{i}k)^2\right)\mathrm{erfc}\left(\frac{\ell_G}{2}(k_n + \mathrm{i}k)\right)\right\}, \tag{B 10}$$

where $\mathrm{erfi}(x) = \mathrm{i}\,\mathrm{erf}(\mathrm{i}x)$ is a real-valued function. It is straightforward to show that for large n, $\mathscr{I}_n \propto 1/n^2$, so that the sum in β_r converges. Substituting (B 9) and (B 10) into (B 6) gives

$$\begin{aligned}\frac{\beta_r}{\omega} = \frac{(\sigma/\ell_G)^2(k\ell_G)^2}{2\cosh^2 kh}&\left\{\frac{2 + \frac{1}{2}\sqrt{\pi}k\ell_G\exp[-(k\ell_G)^2]\mathrm{erfi}(k\ell_G)}{\omega^2 h/g + \sinh^2 kh}\right. \\ &\left. + \sum_{n=1}^{\infty}\frac{2 - k_n\ell_G\sqrt{\pi}\mathrm{Re}\{\exp(\frac{1}{4}\ell_G^2(k_n + \mathrm{i}k)^2)\mathrm{erfc}(\frac{1}{2}\ell_G(k_n + \mathrm{i}k))\}}{\omega^2 h/g - \sin^2 k_n h}\right\}.\end{aligned} \tag{B 11}$$

Substituting the imaginary part of (B 9) into (B 7) yields

$$\frac{\beta_i}{\omega} = \frac{(\sigma k)^2}{\cosh^2 kh\left(\omega^2 h/g + \sinh^2 kh\right)}\frac{\sqrt{\pi}k\ell_G}{4}\left(1 + \exp[-(k\ell_G)^2]\right). \tag{B 12}$$

Substituting (B 12) into (3.3) yields the same localization length as found using (3.6) from the Fourier transform of $\gamma(\xi)$.

As a final note, it can be shown that for any continuous $\gamma(\xi)$, β_i and the localization distance are finite. However, β_r is finite only if the second derivative of $\gamma(\xi)$ is finite at the origin. This is associated with the logarithmic singularity of $\mathscr{G}\left(|\xi|\right)$ at the source $\xi = 0$. A discontinuity in $\gamma'(\xi)$ at $\xi = 0$ implies $\gamma''(\xi) \propto \delta(\xi)$, which gives rise to an unbounded integral in (2.37).

REFERENCES

Ablowitz, M. J. & Segur, H. 1981 *Solitons and the Inverse Scattering Transform.* SIAM.

Anderson, P. A. 1958 Absence of diffusion in certain random lattices. *Phys. Rev.* **109**, 1492–1505.

Asch, M., Kohler, W., Papanicolaou, G. C., Postel, M. & White, B. 1991 Frequency content of randomly scattered signals. *SIAM Rev.* **33**, 519–625.

Bass, F. G., Kivshar, Y. S., Konotop, V. V. & Sinitsyn, Y. A. 1988 Dynamics of solitons under random perturbations. *Phys. Rep.* **157**, 63–181.

Belzons, M., Guazzelli, E. & Parodi, O. 1988 Gravity waves on a rough bottom: experimental evidence of one-dimensional localization. *J. Fluid Mech.* **186**, 539–558.

Bronski, J. C. 1998 Nonlinear wave propagation in a disordered medium. *J. Statist. Phys.* **92**, 995–1015.

Chen, K. K. & Soong, T. T. 1971 Covariance properties of waves propagating in a random medium. *J. Acoust. Soc. Amr.* **49**, 1639–1642.

Chernov, L. A. 1967 *Wave Propagation in a Random Medium.* Dover.

Devillard, P., Dunlop, F. & Souillard, B. 1988 Localization of gravity waves on a channel with a random bottom. *J. Fluid Mech.* **186**, 521–538.

Devillard, P. & Souillard, B. 1986 Polynomially decaying transmission for the nonlinear Schrödinger equation in a random medium. *J. Statist. Phys.* **43**, 423–439.

Doucot, B. & Rammal, R. 1987 Anderson localization in nonlinear random media. *Europhys. Lett.* **3**, 969–974.

Elter, J. F. & Molyneux, J. E. 1972 The long-distance propagation of shallow water waves over an ocean of random depth. *J. Fluid Mech.* 53, 1–15.

Fabrikant, A. & Stepanyants, Y. A. 1998 *Propagation of Waves in Shear Flows.* World Scientific.

Frisch, U. 1968 Wave propagation in random media. In *Probabilistic Methods in Applied Mathematics* (ed. A. T. Bharucha-Reid), vol. 1, pp. 75–198. Academic.

Garnier, J. 1998 Asymptotic transmission of solitons through random media. *SIAM J. Appl. Maths* **58**, 1969–1995.

Garnier, J. 2001*a* Exponential localization versus soliton propagation. *J. Statist. Phys.* **105**, 789–833.

Garnier, J. 2001*b* Solitons in random media with long-range correlation. *Waves in Random Media* **11**, 149–162.

Gredeskul, S. A. & Kivshar, Y. S. 1992 Propagation and scattering of nonlinear waves in disordered systems. *Phys. Rep.* **216**, 1–61.

Hasselman, K. 1966 Feynman diagrams and interaction rules of wave-wave scattering processes. *Rev. Geophys.* **4**, 1–32.

Howe, M. S. 1971 On wave scattering by random inhomogeneities, with application to the theory of weak bores. *J. Fluid Mech.* **45**, 785–804.

Ishimaru, A. 1997 *Wave Propagation and Scattering in Random Media.* IEEE Press.

Johnson, R. S. 1997 *A Modern Introduction to the Mathematical Theory of Water Waves.* Cambridge University Press.

Karal, F. C. & Keller, J. B. 1964 Elastic, electromagnetic and other waves in a random medium. *J. Math. Phys.* **5**, 537–547.

Keller J. B. 1964 Stochastic equations and wave propagation in random media. In *Proc. 16th Symp. Appl. Maths* (ed. R. Bellman), pp. 145–170. Am. Math. Soc.

Kivshar, Y. S., Gredeskul, S. A., Sanchez, A. & Vazquez, L. 1990 Localization decay induced by strong nonlinearity in disordered systems. *Phys. Rev. Lett.* **64**, 1693–1696.

Knapp, R. 1995 Transmission of solitons through random media. *Physica* D **85**, 496–508.

Knapp, R., Papanicolaou, G. & White, B. 1989 Nonlinearity and localization in one-dimensional random media. In *Disorder and Nonlinearity* (ed. A. R. Bishop, D. K. Campbell & S. Pnevmatikos), pp. 2–26. Springer.

Knapp, R., Papanicolaou, G. & White, B. 1991 Transmission of waves by a nonlinear random medium. *J. Statist. Phys.* **63**, 567–583.

Long, R. B. 1973 Scattering of surface waves by an irregular bottom. *J. Geophys. Res.* **78**, 7861–7870.

Mei, C. C. 1989 *Applied dynamics of Ocean Surface Waves*. World Scientific.

Mei, C. C. & Pihl, J. H. 2002 Localization of weakly nonlinear dispersive waves in a random medium. *Proc. R. Soc. Lond.* A **458**, 119–134.

Nachbin, A. 1995 The localization length of randomly scattered water waves. *J. Fluid Mech.* **296**, 353–372.

Nachbin, A. & Papanicolaou, G. C. 1992 Water waves in shallow channels of rapidly varying depth. *J. Fluid Mech.* **241**, 311–332.

Pelinovsky, E., Razin, A. & Sasorova, E. V. 1998 Berkhoff approximation in a problem on surface gravity wave propagation in a basin with bottom irregularities. *Waves in Random Media* **8**, 255–258.

Rosales, R. R. & Papanicolaou, G. C. 1983 Gravity waves in a channel with a rough bottom. *Stud. Appl. Maths* **68**, 89–102.

Satsuma, J. & Yajima, N. 1974 Initial value problems of one-dimensional self-modulation of nonlinear waves in dispersive media. *Suppl. Prog. Theor. Phys.* **55**, 284–306.

Sheng, P. 1990 *Scattering and Localization of Classical Waves in Random Media*. World Scientific.

Sheng, P. 1995 *Introduction to Wave Scattering, Localization, and Mesoscopic Phenomena*. Academic.

Soong, T. T. 1973 *Random Differential Equations in Science and Engineering*. Academic.

Stepaniants, A. 2001 Diffusion and localization of surface gravity waves over irregular bathymetry. *Phys. Rev.* E **63**, 031202/1–11.

Yue, D. K.-P. & Mei, C. C. 1980 Forward diffraction of stokes waves by a thin wedge. *J. Fluid Mech.* **99**, 33–52.